A generalization of the integer linear infeasibility problem

Akimichi Takemura
Department of Mathematical Informatics
University of Tokyo
Bunkyo, Tokyo, Japan
takemura@stat.t.u-tokyo.ac.jp

Ruriko Yoshida
Department of Statistics
University of Kentucky
Lexington, KY USA
ruriko@ms.uky.edu

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Abstract

Does a given system of linear equations $A\boldsymbol{x} = \boldsymbol{b}$ have a nonnegative integer solution? This is a fundamental question in many areas, such as operations research, number theory, and statistics. In terms of optimization, this is called an *integer feasibility problem*. A *generalized integer feasibility problem* is to find $\boldsymbol{b}$ such that there does not exist a nonnegative integral solution in the system with a given A. One such problem is the well-known *Frobenius problem*. In this paper we study the generalized integer feasibility problem and also the multi-dimensional Frobenius problem. To study a family of systems with no nonnegative integer solution, we focus on a commutative semigroup generated by a finite subset of $\mathbb{Z}^d$ and its saturation. An element in the difference of the semigroup and its saturation is called a "hole". We show the necessary and sufficient conditions for the finiteness of the set of holes. Also we define fundamental holes and saturation points of a commutative semigroup. Then, we show the simultaneous finiteness of the set of holes, the set of non-saturation points, and the set of generators for saturation points. As examples we consider some three- and four-way contingency tables from statistics and apply our results to them. Then we will discuss the time complexities of our algorithms.

Key words and phrases: contingency tables, data security, Frobenius problem, indispensable move, Markov basis, monoid, Hilbert basis, linear integer feasibility problem, saturation, semigroup

1 Introduction

Consider the following system of linear equations and inequalities:

$$A\boldsymbol{x} = \boldsymbol{b}, \quad \boldsymbol{x} \geq 0, \tag{1}$$

where $A \in \mathbb{Z}^{d\times n}$ and $\boldsymbol{b} \in \mathbb{Z}^d$. Suppose the solution set $\{x \in \mathbb{R}^n : A\boldsymbol{x} = \boldsymbol{b}, \boldsymbol{x} \geq 0\} \neq \emptyset$. The *linear integer feasibility problem* is to ask whether the system in (1) has an integral solution or

not. A *generalized integer feasibility problem* is to find all $\boldsymbol{b}$ such that there does not exist a nonnegative integral solution in the system with a given A. Note that there exists an integral solution for the system in (1) if and only if $\boldsymbol{b}$ is in the *semigroup* generated by the column vectors of A. From this, we can write this problem as follows.

Problem 1.1. *Let $\boldsymbol{a}_1, \ldots, \boldsymbol{a}_n \in \mathbb{Z}^d$ be columns of A and*

$$Q = Q(A) = \{\boldsymbol{a}_1 x_1 + \cdots + \boldsymbol{a}_n x_n : x_1, \ldots, x_n \in \mathbb{Z}_+\} \tag{2}$$

be the set of all nonnegative integer combinations of $\boldsymbol{a}_1, \ldots, \boldsymbol{a}_n$ or in other words the semigroup *generated by $\boldsymbol{a}_1, \ldots, \boldsymbol{a}_n$. Compute a finite representation of all vectors of Q.*

Barvinok and Woods (2003) introduced an algorithm to encode all vectors in the semigroup Q into a generating function as a *short rational generating function* in polynomial time when d and n are fixed. Therefore, using this algorithm one can compute a finite representation of all vectors not in Q in polynomial time if we fix d and n. However, their algorithm is yet technically difficult to implement so that we do not know whether it is practical or not. Modifying Problem 1.1, in this paper, we would like to solve the following problem.

Problem 1.2. *Let Q be defined as in Problem 1.1. Decide whether there is a finite number of integral vectors not in Q but in its saturation.*

In other words, for fixed A, decide whether there is a finite number of integral vectors $\boldsymbol{b} \in \mathbb{Z}^d$ such that the system in (1) has a nonnegative rational solution but not a nonnegative integral solution.

Intensive research has been carried out on integer feasibility problems. In 1972, Karp (1972) showed that solving the integer linear feasibility problem is NP hard. In the 1980's, H.W. Lenstra, Jr. developed an algorithm to *detect* integer solutions in the system (1) using the LLL-algorithm [Grötschel et al. (1993); Lenstra (1983)]. Lenstra also showed that integer programming problems with a fixed number of variables can be solved in time polynomial in the input size. The algorithm was actually developed in order to prove that the integer feasibility problem can be solved in polynomial time if the dimension is fixed. A later algorithm of similar structure, by Lovász and Scarf (1992), was implemented by Cook et al. (1993). In addition, Aardal and collaborators [Aardal and Lenstra (2002); Aardal et al. (2002, 2000)] have used the LLL-procedure to rewrite a system of linear equations into an equivalent system that was easier to solve with the branch-and-bound method for testing integer feasibility. In the 1990's, based on work by the geometers Brion, Khovanski, Lawrence, and Pukhlikov, Barvinok discovered an algorithm to *count* integer points in rational polytopes, and this algorithm also runs in polynomial time if we fix the dimension [Barvinok (1994); Barvinok and Pommersheim (1999)]. The idea of the algorithm is to encode all the integer solutions for the system in (1) into a rational generating function.

In recent years, the generalized integer linear feasibility problem has found applications in many research areas, such as number theory and statistics. One such problem is the well-known

Frobenius problem, that is, for $d = 1$ and relatively prime positive integers $a_1, \ldots, a_n$, it is to find the biggest positive integer b such that there does not exist an integral solution in (1) [Aardal and Lenstra (2002)]. Equivalently, it is to find the smallest positive integer b' such that there exists an integral solution with $b = b' + \bar{b}$ for any $\bar{b} \in \mathbb{Z}_+$ in (1). Since Georg Frobenius focused on this problem, it attracted substantial attention over more than a hundred years (see [Alfonsin (2006)] for a nice survey). We can generalize the Frobenius problem to the multi-dimensional case. Let $\boldsymbol{a}_1, \ldots, \boldsymbol{a}_n \in \mathbb{Z}^d$ such that the lattice L generated by them is $\mathbb{Z}^d$. Let $K = \text{cone}(\boldsymbol{a}_1, \ldots, \boldsymbol{a}_n)$ be the cone generated by $\boldsymbol{a}_1, \ldots, \boldsymbol{a}_n$ and let Q in (2) be the semigroup generated by $\boldsymbol{a}_1, \ldots, \boldsymbol{a}_n$. Let $S = \{\boldsymbol{b} \in Q : \boldsymbol{b} + (K \cap L) \subset Q\}$. In [Miller and Sturmfels (2005)], a vector $\boldsymbol{b} \in S$ is called a *saturation point* in Q. We ask to find "minimal" elements of S. In the multi-dimensional version of the Frobenius problem, the notion of minimality can be defined in several ways. We present three definitions of minimality and show finiteness results of the set of the minimal elements of S for each definition.

In statistics, one can find an application in the data security problem of *multi-way contingency tables* [Dobra et al. (2003)]. The *3-dimensional integer planar transportation problem* (3-DIPTP) is an integer feasibility problem which asks whether there exists a three dimensional contingency table with the given 2-marginals or not. (In graph theory, a graph is called *planar* if it can be drawn in a plane without graph edges crossing.) For more details on the 3-DIPTP, see [Cox (2002)]. Vlach (1986) provides an excellent summary of attempts on 3-DIPTP.

The linear integer feasibility problem is also closely related to the theory of Markov bases [Diaconis and Sturmfels (1998)] for sampling contingency tables with given marginals by Markov chain Monte Carlo methods. The notion of indispensable moves of Markov bases was defined in [Takemura and Aoki (2004)] and further studied in [Ohsugi and Hibi (2005)]. Recently Ohsugi and Hibi (2006) gave a simple explicit method to construct infeasible equations of (1) from non-squarefree indispensable moves of Markov bases. One finds more details in a discussion of three-way tables in Section 5.

In Section 2 we define saturation points and then we will state our main theorem, Theorem 2.5, which shows the simultaneous finiteness of the set of *holes*, which is the difference between the semigroup and its saturation, the set of *non-saturation points* of the semigroup, and the set of generators for saturation points. In Section 3, we show the necessary and sufficient condition for the finiteness of the set of holes. Section 4 shows a proof of Theorem 2.5. Section 5 contains various computational results for three- and four-way contingency tables. Section 6 will discuss that (1) solving Problem 1.1, (2) solving Problem 1.2, (3) computing the set of *holes*, and (4) computing the set of *fundamental holes* are polynomial time in fixed d and n.

2 Notation and the main theorem

In this section we will remind the reader of some definitions and we will set appropriate notation. We follow the notation in Chapter 7 of [Miller and Sturmfels (2005)] and [Sturmfels (1996)]. Let

$A \in \mathbb{Z}^{d\times n}$ and let $\boldsymbol{a}_1, \ldots, \boldsymbol{a}_n$ denote the columns of A. Let $\mathbb{N} = \mathbb{Z}_+ = \{0, 1, \ldots\}$.

Definition 2.1. *Let Q in (2) be the semigroup generated by $\boldsymbol{a}_1, \ldots, \boldsymbol{a}_n$, let $K = \mathrm{cone}(\boldsymbol{a}_1, \ldots, \boldsymbol{a}_n)$ be the cone generated by $\boldsymbol{a}_1, \ldots, \boldsymbol{a}_n$, and let L be the lattice generated by $\boldsymbol{a}_1, \ldots, \boldsymbol{a}_n$. Then the semigroup $Q_{\mathrm{sat}} = K \cap L$ is called the* saturation *of the semigroup Q. $Q \subset Q_{\mathrm{sat}}$ and we call Q* saturated *if $Q = Q_{\mathrm{sat}}$ (also this is called* normal*). $H = Q_{\mathrm{sat}} \setminus Q$ is the set of holes. $\boldsymbol{a} \in Q$ is called a saturation point if $\boldsymbol{a} + Q_{\mathrm{sat}} \subset Q$.*

We assume $L = \mathbb{Z}^d$ without loss of generality for our theoretical developments in Sections 3 and 4. This is for convenience in working with the Hilbert basis of K. The following is a list of some notations through this paper:

$$
\begin{aligned}
K &= A\mathbb{R}^n_+ = \{\boldsymbol{a}_1 x_1 + \cdots + \boldsymbol{a}_n x_n : x_1, \ldots, x_n \in \mathbb{R}_+\} \\
Q_{\mathrm{sat}} &= K \cap L = \text{saturation of } A \supset Q \\
H &= Q_{\mathrm{sat}} \setminus Q = \text{holes in } Q_{\mathrm{sat}} \\
S &= \{\boldsymbol{a} \in Q : \boldsymbol{a} + Q_{\mathrm{sat}} \subset Q\} = \text{saturation points of } Q \\
\bar{S} &= Q \setminus S = \text{non-saturation points of } Q
\end{aligned}
$$

We assume that there exists $\boldsymbol{c} \in \mathbb{Q}^d$ such that $\boldsymbol{c} \cdot \boldsymbol{a}_i > 0$ for $i = 1, \ldots, n$, where $\cdot$ is the standard inner product. Under this assumption K and Q are *pointed* and S is non-empty by Problem 7.15 of [Miller and Sturmfels (2005)]. Q_{sat} is partitioned as

$$Q_{\mathrm{sat}} = H \cup \bar{S} \cup S = H \cup Q.$$

Equivalently

$$S \subset Q \subset Q_{\mathrm{sat}} \tag{3}$$

and the differences of these two inclusions are $\bar{S}$ and H, respectively.

If Q is saturated (equivalently $H = \emptyset$), then $0 \in S$ and $S = Q$, because $Q = 0 + Q \subset S + Q \subset S$. Therefore $S = Q = Q_{\mathrm{sat}}$ in (3). Similarly if $S = Q$, then $0 \in S$ and $Q_{\mathrm{sat}} \subset Q$, implying Q is saturated. From this consideration it follows that either $S = Q = Q_{\mathrm{sat}}$ or the two inclusions in (3) are simultaneously strict.

We now consider three different notions of the minimality of saturation points, i.e., points of S which are minimal with respect to S, Q, and Q_{sat}. We call $\boldsymbol{a} \in S$ an S-minimal (a Q-minimal, a Q_{sat}-minimal, resp.) if there exists no other $\boldsymbol{b} \in S$, $\boldsymbol{b} \neq \boldsymbol{a}$, such that $\boldsymbol{a} - \boldsymbol{b} \in S$ (Q, Q_{sat}, resp.). More formally $\boldsymbol{a} \in S$ is

a) an S-minimal saturation point if $(\boldsymbol{a} + (-(S \cup \{0\}))) \cap S = \{\boldsymbol{a}\}$,

b) a Q-minimal saturation point if $(\boldsymbol{a} + (-Q)) \cap S = \{\boldsymbol{a}\}$,

c) a Q_{sat}-minimal saturation point if $(\boldsymbol{a} + (-Q_{\mathrm{sat}})) \cap S = \{\boldsymbol{a}\}$.

Let $\min(S;S)$ denote the set of S-minimal saturation points, $\min(S;Q)$ the set of Q-minimal saturation points, and $\min(S;Q_{\text{sat}})$ the set of Q_{sat}-minimal saturation points. Because of the inclusion (3), it follows that

$$\min(S;Q_{\text{sat}}) \subset \min(S;Q) \subset \min(S;S). \tag{4}$$

If $\boldsymbol{a} \in H$, then for any $\boldsymbol{b} \in Q$, either $\boldsymbol{a}-\boldsymbol{b} \notin Q_{\text{sat}}$ or $\boldsymbol{a}-\boldsymbol{b} \in H$. This is because if $\boldsymbol{a}-\boldsymbol{b} \in Q_{\text{sat}}$ and $\boldsymbol{a}-\boldsymbol{b} \notin H$, then $\boldsymbol{a}-\boldsymbol{b} \in Q$, and hence $\boldsymbol{a} = \boldsymbol{b} + (\boldsymbol{a}-\boldsymbol{b}) \in Q$, which contradicts $\boldsymbol{a} \in H$. This relation can be expressed as

$$Q_{\text{sat}} \cap (H + (-Q)) = H.$$

This relation suggests the following definition.

Definition 2.2. *We call $\boldsymbol{a} \in Q_{\text{sat}}$, $\boldsymbol{a} \neq 0$, a* fundamental hole *if*

$$Q_{\text{sat}} \cap (\boldsymbol{a} + (-Q)) = \{\boldsymbol{a}\}.$$

Let H_0 be the set of fundamental holes.

Example 2.3. *Consider the one-dimensional example $A = (3\ 5\ 7)$ with $L = \mathbb{Z}$. $Q_{\text{sat}} = \{0,1,\ldots\}$, $Q = \{0,3,5,6,7,\ldots\}$, $-Q = \{0,-3,-5,-6,-7,\ldots\}$, $H = \{1,2,4\}$, $S = \{5,6,7,\ldots\}$ and $\bar{S} = \{0,3\}$. Among the 3 holes, 1 and 2 are fundamental. For example, $2 \in H$ is fundamental because*

$$\{0,1,\ldots\} \cap \{2,-1,-3,-4,-5,\ldots\} = \{2\}.$$

On the other hand $4 \in H$ is not fundamental because

$$\{0,1,\ldots\} \cap \{4,1,-1,-2,-3,\ldots\} = \{4,1\}.$$

If $0 \neq \boldsymbol{a} \in Q$, then $Q_{\text{sat}} \cap (\boldsymbol{a} + (-Q)) \supset \{\boldsymbol{a}, 0\}$ and $\boldsymbol{a}$ is not a fundamental hole. This implies that a fundamental hole is a hole. For every non-fundamental hole $\boldsymbol{x}$, there exists $\boldsymbol{y} \in H$ such that $0 \neq \boldsymbol{x} - \boldsymbol{y} \in Q$. If $\boldsymbol{y}$ is not fundamental we can repeat this procedure. Since the procedure has to stop in finite number of steps, it follows that every non-fundamental hole $\boldsymbol{x}$ can be written as

$$\boldsymbol{x} = \boldsymbol{y} + \boldsymbol{a}, \qquad \boldsymbol{y} \in H_0, \quad \boldsymbol{a} \in Q,\ \boldsymbol{a} \neq 0. \tag{5}$$

We also focus on a *Hilbert basis* of a cone K and in the next section we will show a relation between the set of holes H and the *minimal Hilbert basis* of a pointed cone K.

Definition 2.4. *We call a finite subset $B \subset K \cap \mathbb{Z}^d$ a* Hilbert basis *of a cone K if any integral point in K can be written as a nonnegative integral linear combination of elements in B. If B is minimal in terms of inclusion then we call it a* minimal Hilbert basis *of K.*

Note that there exists a Hilbert basis for any rational polyhedral cone and also if a cone is pointed then there exists a unique minimal Hilbert basis [see Schrijver (1986) for more details].

Now we will present our main theorem of this paper and then we will present small examples to demonstrate the theorem. In the theorem, $\mathrm{cone}(S)$ denotes the set of finite nonnegative real combinations of elements of S and "rational polyhedral cone" is a closed cone defined by rational linear weak inequalities (inequalities that permit the equality case). One can find a proof of this theorem in Section 4.

Theorem 2.5. *The following statements are equivalent.*

1. $\min(S; S)$ *is finite.*
2. $\mathrm{cone}(S)$ *is a rational polyhedral cone.*
3. *There is some* $s \in S$ *on every extreme ray of* K*.*
4. H *is finite.*
5. $\bar{S}$ *is finite.*

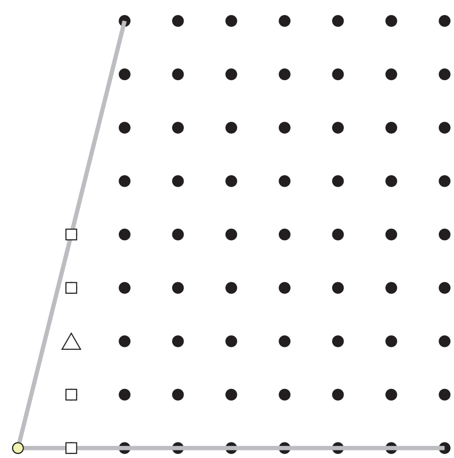

Figure 1: White circles represent non-saturation points, a triangle represents a hole, white squares represent S-minimal saturation points, and black circles represent non S-minimal saturation points in the semigroup in Example 2.6.

Example 2.6. *Let* A *be an integral matrix such that*

$$A = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 3 & 4 \end{pmatrix}.$$

The set of holes H *consists of only one element* $\{(1,2)^t\}$*.* $\bar{S} = \{(0,0)^t\}$*.* $\min(S;S) = \{(1,0)^t, (1,1)^t, (1,3)^t, (1,4)^t\}$*. Thus,* H*,* $\bar{S}$*, and* $\min(S;S)$ *are all finite.*

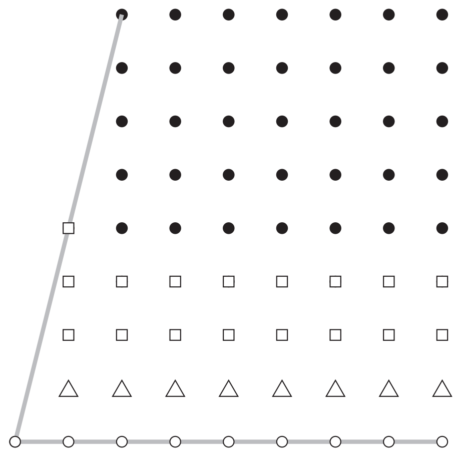

Figure 2: White circles represent non-saturation points, triangles represent holes, white squares represent S-minimal saturation points, and black circles represent non S-minimal saturation points in the semigroup in Example 2.7.

Example 2.7. *Let A be an integral matrix such that*

$$A = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 2 & 3 & 4 \end{pmatrix}.$$

The set of holes H are the elements $\{(k,1) : k \in \mathbb{Z},\, k \geq 1\}$. $\bar{S} = \{(i,0)^t : i \in \mathbb{Z},\, i \geq 0\}$, and $\min(S;S) = \{(k,j)^t : k \in \mathbb{Z},\, k \geq 1,\, 2 \leq j \leq 3\} \cup \{(1,4)\}$. *Thus, H, $\bar{S}$, and* $\min(S;S)$ *are all infinite. However,* $\min(S;Q) = \{(1,2)^t,\, (1,3)^t,\, (1,4)^t\}$ *is finite.*

3 Necessary and sufficient condition of finiteness of a set of holes

In this section we give a necessary and sufficient condition of finiteness of the set of holes H. Firstly we will show the necessary and sufficient condition in terms of the set of fundamental holes H_0. Then we generalize the statement, such that it is stated in terms of the minimal Hilbert basis of K. Ezra Miller has kindly pointed out to the authors that many of our results can be proved more succinctly by appropriate algebraic methods. However for the sake of self-contained presentation we provide our own proofs and summarize his comments in Remark 3.2 and Remark 3.6 below.

First we show that the set of fundamental holes, H_0, is finite.

Proposition 3.1. *H_0 is finite.*

Proof. Every $\boldsymbol{a} \in Q_{\mathrm{sat}}$ can be written as

$$\boldsymbol{a} = c_1\boldsymbol{a}_1 + \cdots + c_n\boldsymbol{a}_n, \tag{6}$$

where c_i's are nonnegative rational numbers. (If $\boldsymbol{a} \in H$, then at least one c_i is not integral.) If $c_1 > 1$, then $\boldsymbol{a}$ can be written as

$$\boldsymbol{a} = \{(c_1 - \lfloor c_1 \rfloor)\boldsymbol{a}_1 + \cdots + c_n \boldsymbol{a}_n\} + \lfloor c_1 \rfloor \boldsymbol{a}_1 = \tilde{\boldsymbol{a}} + \lfloor c_1 \rfloor \boldsymbol{a}_1,$$

and $\tilde{\boldsymbol{a}} = \boldsymbol{a} - \lfloor c_1 \rfloor \boldsymbol{a}_1$. Therefore

$$Q_{\mathrm{sat}} \cap (\boldsymbol{a} + (-Q)) \supset \{\boldsymbol{a}, \tilde{\boldsymbol{a}}\}$$

and $\boldsymbol{a}$ is not a fundamental hole. In this argument we can replace c_1 with any c_i, $i \geq 2$. This shows each fundamental hole has an expression (6), where $0 \leq c_i \leq 1$, $i = 1, \ldots, n$. However fundamental holes belong to a compact set. Since the lattice points in a compact set are finite, H_0 is finite. □

Remark 3.2. *For any field k, consider the semigroup rings $k[Q]$ and $k[Q_{\mathrm{sat}}]$. Define $M = k[Q_{\mathrm{sat}}]/k[Q]$, which is finitely generated as a module over $k[Q]$. H_0 is the set of degrees for the minimal generators of M and therefore H_0 is finite.*

Let $H_0 = \{\boldsymbol{y}_1, \ldots, \boldsymbol{y}_m\}$. Now for each $\boldsymbol{y}_h \in H_0$ and each $\boldsymbol{a}_i$ define $\bar{\lambda}_{hi}$ as follows. If there exists some $\lambda \in \mathbb{Z}$ such that $\boldsymbol{y}_h + \lambda \boldsymbol{a}_i \in Q$, let

$$\bar{\lambda}_{hi} = \min\{\lambda \in \mathbb{Z} \mid \boldsymbol{y}_h + \lambda \boldsymbol{a}_i \in Q\}. \tag{7}$$

Otherwise define $\bar{\lambda}_{hi} = \infty$. Note that $\bar{\lambda}_{hi} > 0$ because $\boldsymbol{y}_h$ is a hole. Then we have the following result:

Theorem 3.3. *H is finite if and only if $\bar{\lambda}_{hi} < \infty$ for all $h = 1, \ldots, m$ and all $i = 1, \ldots, n$.*

Proof. For one direction, assume that $\bar{\lambda}_{hi} = \infty$ for some h and i. Then $\boldsymbol{y}_h + \lambda \boldsymbol{a}_i$, $\lambda = 1, 2, \ldots,$ all belong to Q_{sat} but do not belong to Q. Therefore they are holes. Hence H is infinite.

For the other direction, assume that $\bar{\lambda}_{hi} < \infty$ for all $h = 1, \ldots, m$ and all $i = 1, \ldots, n$. By (5), each hole can be written as

$$\boldsymbol{x} = \boldsymbol{y}_h + \sum_{i=1}^{n} \lambda_i \boldsymbol{a}_i$$

for some h and $\lambda_i \in \mathbb{N}$, $i = 1, \ldots, n$. Now suppose that $\lambda_i \geq \bar{\lambda}_{hi}$ for some i. Then

$$\boldsymbol{y}_h + \lambda_i \boldsymbol{a}_i \in Q$$

and

$$\boldsymbol{x} = \boldsymbol{y}_h + \lambda_i \boldsymbol{a}_i + \sum_{j \neq i} \lambda_j \boldsymbol{a}_j \in Q,$$

which contradicts that $\boldsymbol{x}$ is a hole. Therefore if $\boldsymbol{x}$ is a hole, then $\lambda_i < \bar{\lambda}_{hi}$ for all i. Then

$$H \subset \{\boldsymbol{y}_h + \sum_{i=1}^{n} \lambda_{hi} \boldsymbol{a}_i \mid h = 1, \ldots, m,\ 0 \leq \lambda_{hi} < \bar{\lambda}_{hi}\}.$$

The right-hand side is finite. □

Remark 3.4. *There are several remarks to make. For each* $1 \leq i \leq n$*, let*

$$\tilde{Q}_{(i)} = \{\sum_{j \neq i} \lambda_j \boldsymbol{a}_j \mid \lambda_j \in \mathbb{N},\ j \neq i\}$$

be the semigroup spanned by $\boldsymbol{a}_j, j \neq i$*. Furthermore write*

$$\bar{Q}_{(i)} = \mathbb{Z}\boldsymbol{a}_i + \tilde{Q}_{(i)}.$$

For each h *and* i*,* $\bar{\lambda}_{hi}$ *is finite if and only if* $\boldsymbol{y}_h \in \bar{Q}_{(i)}$*. Since* $\boldsymbol{y}_h$ *is a hole, actually we only need to check*

$$\boldsymbol{y}_h \in (-\mathbb{N}\boldsymbol{a}_i) + \tilde{Q}_{(i)}.$$

But $(-\mathbb{N}\boldsymbol{a}_i) + \tilde{Q}_{(i)}$ *is another semigroup, where* $\boldsymbol{a}_i$ *in* A *is replaced by* $-\boldsymbol{a}_i$*. Therefore this problem is a standard membership problem in a semigroup.*

Also we only need to check i such that $\boldsymbol{a}_i$ is on an extreme ray. By a slight abuse of terminology, we simply say that $\boldsymbol{a}_i$ is an extreme ray if $\boldsymbol{a}_i$ generates an extreme ray of K. If there are multiple columns of A on the same extreme ray, for definiteness we choose the smallest one, although we can choose any one of them. Assume, without loss of generality, that $\{\boldsymbol{a}_1, \ldots, \boldsymbol{a}_k\}$, $k \leq n$, is the set of the extreme rays. The following corollary says that we only need to consider $i \leq k$.

Corollary 3.5. H *is finite if and only if* $\bar{\lambda}_{hi} < \infty$ *for all* $h = 1, \ldots, m$ *and all* $i = 1, \ldots, k$*.*

Proof. The first direction is the same as above.

For the converse direction, we show that if $\bar{\lambda}_{hi} < \infty$, $1 \leq h \leq m$, $1 \leq i \leq k$, then $\bar{\lambda}_{hi} < \infty$, $1 \leq h \leq m$, $k+1 \leq i \leq n$. Now any non-extreme ray $\boldsymbol{a}_i$, $i \geq k+1$, can be written as a nonnegative rational combination of extreme rays:

$$\boldsymbol{a}_i = \sum_{j=1}^{k} q_{ij} \boldsymbol{a}_j, \qquad i \geq k+1. \tag{8}$$

Let $\bar{q}_i > 0$ denote the l.c.m. of the denominators of $q_{i1}, \ldots, q_{ik}$. Then multiplying both sides by $\bar{q}_i$, we have

$$\bar{q}_i \boldsymbol{a}_i = \sum_{j=1}^{k} (\bar{q}_i q_{ij}) \boldsymbol{a}_j, \quad \bar{q}_i q_{ij} \in \mathbb{N}.$$

Also note that there is at least one $q_{ij} > 0$, say q_{ij_0}. Consider $\bar{q}_i \boldsymbol{a}_i, 2\bar{q}_i \boldsymbol{a}_i, 3\bar{q}_i \boldsymbol{a}_i, \ldots$. Take $\lambda \in \mathbb{N}$ such that

$$\lambda \bar{q}_i q_{ij_0} \geq \bar{\lambda}_{hj_0}.$$

Then by (5)

$$\boldsymbol{y}_h + \lambda \bar{q}_i \boldsymbol{a}_i = \boldsymbol{y}_h + \lambda \bar{q}_i q_{ij_0} \boldsymbol{a}_{j_0} + \sum_{j \neq j_0}^{k} \lambda \bar{q}_i q_{ij} \boldsymbol{a}_j \in Q.$$

$\square$

Remark 3.6. *Let $k[Q]$, $k[Q_{\text{sat}}]$ and M be defined as in Remark 3.2. The number of points in H is the k-vector space dimension of M. H is finite if and only if M is Artinian, which proves Theorem 3.3. Let $k[Q_{\text{rays}}]$ denote the monoid generated by the smallest lattice points in Q on the real extreme rays of Q. Then $k[Q]$ is itself finitely generated as a module over the $k[Q_{\text{rays}}]$. This proves Corollary 3.5.*

Another important point is that we want to state Theorem 3.3 in terms of Hilbert bases. Let $B = \{\boldsymbol{b}_1, \ldots, \boldsymbol{b}_L\}$ denote the Hilbert basis of K. As above, if $\boldsymbol{b}_l + \lambda \boldsymbol{a}_i \in Q$ for some $\lambda \in \mathbb{Z}$ let

$$\bar{\mu}_{li} = \min\{\lambda \in \mathbb{Z} \mid \boldsymbol{b}_l + \lambda \boldsymbol{a}_i \in Q\}$$

and $\bar{\mu}_{li} = \infty$ otherwise. Then we have the following theorem.

Theorem 3.7. *H is finite if and only if $\bar{\mu}_{li} < \infty$ for all $l = 1, \ldots, L$ and all $i = 1, \ldots, n$.*

Proof. The first direction is the same as the above proofs.

For the converse direction, assume that $\bar{\mu}_{li} < \infty$ for all $l = 1, \ldots, L$ and all $i = 1, \ldots, n$. Let $\boldsymbol{y}_h$ be a fundamental hole. It can be written as a nonnegative integral combination of the elements of the Hilbert basis

$$\boldsymbol{y}_h = \sum_{l=1}^{L} \alpha_{hl} \boldsymbol{b}_l.$$

Let

$$\lambda = \sum_{l=1}^{L} \alpha_{hl} \bar{\mu}_{li}.$$

Then by (5)

$$\begin{aligned} \boldsymbol{y}_h + \lambda \boldsymbol{a}_i &= \sum_{l=1}^{L} \alpha_{hl} \boldsymbol{b}_l + \big(\sum_{l=1}^{L} \alpha_{hl} \bar{\mu}_{li} \big) \boldsymbol{a}_i \\ &= \sum_{l=1}^{L} \alpha_{hl} (\boldsymbol{b}_l + \bar{\mu}_{li} \boldsymbol{a}_i) \in Q. \end{aligned}$$

This implies $\bar{\lambda}_{hi} < \infty$ for all h and i. □

As in Corollary 3.5, it is clear that we only need to check extreme rays among $\boldsymbol{a}_i$'s.

Corollary 3.8. *H is finite if and only if $\bar{\mu}_{li} < \infty$ for all $l = 1, \ldots, L$ and all $i = 1, \ldots, k$.*

Remark 3.9. *In summary, determining finiteness of H is straightforward. We obtain the Hilbert basis B of Q_{sat}. For each $\boldsymbol{b} \in B \setminus Q$ and for each extreme $\boldsymbol{a}_i$, we check*

$$\boldsymbol{b} \in (-\mathbb{N}\boldsymbol{a}_i) + \tilde{Q}_{(i)}.$$

Example 3.10. *Let A be an integral matrix such that*

$$A = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 3 & 4 \end{pmatrix}.$$

Then B consists of 5 elements

$$B = \{\boldsymbol{b}_1 = (1,0)^t, \boldsymbol{b}_2 = (1,1)^t, \boldsymbol{b}_3 = (1,2)^t, \boldsymbol{b}_4 = (1,3)^t, \boldsymbol{b}_5 = (1,4)^t\}.$$

Then we can write $\boldsymbol{b}_3$ as the following:

$$\begin{aligned} (1,2)^t &= -(1,0)^t + 2 \cdot (1,1)^t \\ &= (1,0)^t - (1,1)^t + (1,3)^t \\ &= (1,1)^t - (1,3)^t + (1,4)^t \\ &= 2 \cdot (1,3)^t - (1,4)^t. \end{aligned}$$

Thus, in this case, we have $\bar{\mu}_{3i} = 1$ for each $i = 1, \ldots, 4$ and $\bar{\mu}_{li} = 0$, where $l \neq 3$ and each $i = 1, \ldots, 4$. Thus by Theorem 3.7, the number of elements in H is finite. Note that H consists of only one element $\{\boldsymbol{b}_3 = (1,2)^t\}$.

4 Simultaneous finiteness of holes, non-saturation points, and minimal saturation points

In this section we will show the simultaneous finiteness of holes, non-saturation points, and S-minimal saturation points. As in the previous section let $\{\boldsymbol{a}_1, \ldots, \boldsymbol{a}_k\}$, $k \leq n$, be the set of the extreme rays. First, we will show the following lemmas.

Lemma 4.1. *Suppose that Q is not saturated. $\boldsymbol{a} \in Q$ is a saturation point if and only if $\boldsymbol{a} + \boldsymbol{y} \in Q$ for all fundamental holes $\boldsymbol{y}$.*

Proof. If $\boldsymbol{a} \in Q$ is a saturation point, then $\boldsymbol{a} + \boldsymbol{y} \in Q$ for all $\boldsymbol{y} \in Q_{\mathrm{sat}}$. In particular $\boldsymbol{a} + \boldsymbol{y} \in Q$ for all fundamental holes $\boldsymbol{y}$.

Now suppose that $\boldsymbol{a} \in Q$ is not a saturation point. Then there exists $\boldsymbol{y} \in Q_{\mathrm{sat}}$ such that $\boldsymbol{a} + \boldsymbol{y}$ is a hole. This $\boldsymbol{y}$ has to be a hole, because otherwise $\boldsymbol{a} + \boldsymbol{y} \in Q$. $\boldsymbol{y}$ can be written as $\boldsymbol{y} = \boldsymbol{y}_h + \boldsymbol{b}$ for some fundamental hole $\boldsymbol{y}_h$ and $\boldsymbol{b} \in Q$. Then $\boldsymbol{a} + \boldsymbol{y} = \boldsymbol{a} + \boldsymbol{y}_h + \boldsymbol{b}$ and $\boldsymbol{a} + \boldsymbol{y}_h$ has to be hole. Therefore we have shown that if $\boldsymbol{a}$ is not a saturation point, then $\boldsymbol{a} + \boldsymbol{y}$ is a hole for some fundamental hole $\boldsymbol{y}$. □

Lemma 4.2. *Suppose that Q is not saturated. Consider any column $\boldsymbol{a}_i$ of A. There exists some $n_i \in \mathbb{N}$ such that $n_i \boldsymbol{a}_i \in S$ if and only if $\bar{\lambda}_{hi} < \infty$ in (7) for all $h = 1, \ldots, m$.*

Proof. This follows from Lemma 4.1. If $n_i \boldsymbol{a}_i \in S$, $\bar{\lambda}_{hi} \leq n_i$. For the other direction take $n_i = \max_h \bar{\lambda}_{hi}$. □

Now we consider the following two conditions.

Condition 1 For each $\boldsymbol{a}_i$, there exists $n_i > 0$ such that $n_i \boldsymbol{a}_i \in S$.

Condition 2 For each extreme ray $\boldsymbol{a}_i$, $1 \leq i \leq k$, there exists $n_i > 0$ such that $n_i \boldsymbol{a}_i \in S$.

Proposition 4.3. *Condition* 1*, Condition* 2*, and the finiteness of* H *are equivalent.*

Proof. Condition 1 trivially implies Condition 2. On the other hand suppose that Condition 2 holds. Then each non-extreme $\boldsymbol{a}_i$, $k < i \leq n$, can be written as (8). As above let $\bar{q}_i > 0$ denote the l.c.m. of the denominators of $q_{i1}, \ldots, q_{ik}$ and let $n_i = \bar{q}_i \times n_1 \times \cdots \times n_k$, then $n_i \boldsymbol{a}_i \in S$ and Condition 1 holds.

Now we show the equivalence between the finiteness of H and the other two conditions. Using Lemma 4.2, Condition 1 is equivalent to the condition in Theorem 3.3. Also Condition 2 is equivalent to the condition in Corollary 3.5. □

Now we prove Theorem 2.5.

Proof for Theorem 2.5. 1. $\iff$ 2. : $\min(S;S)$ is an integral generating set of the monoid $S \cup \{0\}$. We then apply Theorem 1.1 (b) of [Hemmecke and Weismantel (2006)] or Theorem 4 in [Jeroslow (1978)].

2. $\iff$ 3. : If $\mathrm{cone}(S)$ is not polyhedral, there must be an extreme ray e of K not in $\mathrm{cone}(S)$, since K is polyhedral. Thus, $e \cap S = \emptyset$.

If $\mathrm{cone}(S)$ is polyhedral, then it is a rational polyhedron and has a finite integral generating set. Thus, by Theorem 8.8 in [Bertsimas and Weismantel (2005)] the polyhedron $\mathrm{cone}(S)$ contains all lattice points from its recession cone K and $(K \setminus \mathrm{cone}(S)) \cap \mathbb{Z}^d$ is finite, which in this case can only happen if $\mathrm{cone}(S) = K$. Thus, there is a point from S on each extreme ray of K.

3. $\iff$ 4. : The statement 3. is equivalent to Condition 2. Thus, the proof follows directly by Proposition 4.3.

4. $\iff$ 5. : Suppose that H is finite. Then by Condition 1, it is easy to see that $\bar{S}$ is contained in a compact set and hence $\bar{S}$ is finite. For the opposite implication, suppose that H is infinite. Since Condition 1 does not hold, there exists some i such that $n\boldsymbol{a}_i \notin S$ for all $n \in \mathbb{N}$. Then $\{\boldsymbol{a}_i, 2\boldsymbol{a}_i, 3\boldsymbol{a}_i, \ldots\} \subset \bar{S}$ and $\bar{S}$ is infinite. □

Now we consider the generators $\min(S;Q)$ and we prove that $\min(S;Q)$ is always finite. Then by (4) $\min(S;Q_{\mathrm{sat}})$ is always finite as well. Note that the multi-dimensional Frobenius problem can be stated as computing the sets $\min(S;Q)$ and $\min(S;Q_{\mathrm{sat}})$.

Proposition 4.4. $\min(S;Q)$ *is finite.*

Proof. Note that Q is a finitely generated monoid. Consider the algebra, $k[Q] := k[t^{\boldsymbol{a}_1}, \cdots, t^{\boldsymbol{a}_n}]$, where k is any algebraic field. Then $k[Q]$ is a finitely generated k-algebra by Proposition 2.5 in [Bruns and Gubeladze (2006)] and therefore a Noetherian ring by a corollary of Hilbert's basis

theorem (Corollary 1.3 in [Eisenbud (1995)]). Since $I_S :=< t^\beta : \beta \in S >$ is an ideal in $k[Q]$, we are done. $\square$

A combinatorial proof of this proposition is given in [Hemmecke et al. (2007)].

Proposition 4.5.

$$\min(S; Q) \subset \min(S; Q_{\text{sat}}) + (H_0 \cup \{0\}). \tag{9}$$

Proof. Let $\boldsymbol{a} \in \min(S; Q)$. We want to show that $\boldsymbol{a}$ can be written as $\boldsymbol{a} = \tilde{\boldsymbol{a}} + \boldsymbol{b}$, where $\tilde{\boldsymbol{a}} \in \min(S; Q_{\text{sat}})$ and $\boldsymbol{b} \in H_0 \cup \{0\}$. If $\boldsymbol{a}$ itself belongs to $\min(S; Q_{\text{sat}})$, then take $\boldsymbol{a} = \tilde{\boldsymbol{a}}$ and $\boldsymbol{b} = 0$. Otherwise, if $\boldsymbol{a} \notin \min(S; Q_{\text{sat}})$, then by definition of Q_{sat}-minimality there exists $\boldsymbol{a}' \in S$ such that $0 \neq \boldsymbol{a} - \boldsymbol{a}' \in Q_{\text{sat}}$. If $\boldsymbol{a}' \notin \min(S; Q_{\text{sat}})$, then we can do the same operation to $\boldsymbol{a}'$. This operation has to stop in finite steps and we arrive at $\tilde{\boldsymbol{a}} \in \min(S; Q_{\text{sat}})$ such that $\boldsymbol{b} = \boldsymbol{a} - \tilde{\boldsymbol{a}} \in Q_{\text{sat}}$. If this $\boldsymbol{b} \notin H_0$, then there exists $\boldsymbol{c} \in Q$, $\boldsymbol{c} \neq 0$, such that $\boldsymbol{b} - \boldsymbol{c} \in Q_{\text{sat}}$. Then

$$\boldsymbol{a} = \tilde{\boldsymbol{a}} + \boldsymbol{b} = \tilde{\boldsymbol{a}} + (\boldsymbol{b} - \boldsymbol{c}) + \boldsymbol{c},$$

where $\tilde{\boldsymbol{a}} \in S$, $\boldsymbol{b} - \boldsymbol{c} \in Q_{\text{sat}}$. Since $S + Q_{\text{sat}} \subset S$, $\tilde{\boldsymbol{a}} + (\boldsymbol{b} - \boldsymbol{c}) \in S$. But this contradicts $\boldsymbol{a} \in \min(S; Q)$. $\square$

5 Applications to contingency tables

An s**-way contingency table of size** $n_1 \times \cdots \times n_s$ is an array of nonnegative integers $v = (v_{i_1,\ldots,i_s})$, $1 \leq i_j \leq n_j$. For $0 \leq r < s$, an r**-marginal** of v is any of the $\binom{s}{r}$ possible r-way tables obtained by summing the entries over all but r indices. In this section we apply our theorem to some examples including $2 \times 2 \times 2 \times 2$ tables with 2-marginals and $2 \times 2 \times 2 \times 2$ tables with three 2-marginals and a 3-marginal ([12][13][14][234]). Also we apply our theorem to three-way contingency tables from [Vlach (1986)]. To compute minimal Hilbert bases of cones, we used `normaliz` [Bruns and Koch (2001)] and to compute each hyperplane representation and vertex representation we used `CDD` [Fukuda (2005)] and `lrs` [Avis (2005)]. Also we used `4ti2` [Hemmecke et al. (2005)] to compute matrix A for the system.

$2 \times 2 \times 2 \times 2$ tables

$2 \times 2 \times 2 \times 2$ tables with 2-marginals

First, we would like to show some simulation results with $2 \times 2 \times 2 \times 2$ tables with 2-marginals, which can be seen as the complete graph with 4 nodes $K4$ and with 2 states on each node. The semigroup of $K4$ has 16 generators $\boldsymbol{a}_1, \ldots, \boldsymbol{a}_{16}$ in $\mathbb{Z}^{24}$ (without removing redundant rows) such that

```
1 0 0 0 1 0 0 0 1 0 0 0 1 0 0 0
0 1 0 0 0 1 0 0 0 1 0 0 0 1 0 0
```

0	0	1	0	0	0	1	0	0	0	1	0	0	0	1	0
0	0	0	1	0	0	0	1	0	0	0	1	0	0	0	1
1	0	1	0	0	0	0	0	1	0	1	0	0	0	0	0
0	1	0	1	0	0	0	0	0	1	0	1	0	0	0	0
0	0	0	0	1	0	1	0	0	0	0	0	1	0	1	0
0	0	0	0	0	1	0	1	0	0	0	0	0	1	0	1
1	0	1	0	1	0	1	0	0	0	0	0	0	0	0	0
0	1	0	1	0	1	0	1	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	1	0	1	0	1	0	1	0
0	0	0	0	0	0	0	0	0	1	0	1	0	1	0	1
1	1	0	0	0	0	0	0	1	1	0	0	0	0	0	0
0	0	1	1	0	0	0	0	0	0	1	1	0	0	0	0
0	0	0	0	1	1	0	0	0	0	0	0	1	1	0	0
0	0	0	0	0	0	1	1	0	0	0	0	0	0	1	1
1	1	0	0	1	1	0	0	0	0	0	0	0	0	0	0
0	0	1	1	0	0	1	1	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	1	1	0	0	1	1	0	0
0	0	0	0	0	0	0	0	0	0	1	1	0	0	1	1
1	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	1	1	1	1	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	1	1	1	1	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	1	1	1	1

Remember that the columns of the given array are the generators of the semigroup. All of these vectors are extreme rays of the cone, which we verified via `cddlib` [Fukuda (2005)]. The Hilbert basis of the cone generated by these 16 vectors contains 17 vectors $\boldsymbol{b}_1, \ldots, \boldsymbol{b}_{17}$. The first 16 vectors are the same as $\boldsymbol{a}_i$, i.e. $\boldsymbol{b}_i = \boldsymbol{a}_i$, $i = 1, \ldots, 16$. The 17-th vector $\boldsymbol{b}_{17}$ is

$$\boldsymbol{b}_{17} = (1\ 1\ \ldots\ 1)^t$$

consisting of all 1's. Thus, $\boldsymbol{b}_{17} \notin Q$. Then we set the 16 systems of linear equations such that:

$$\begin{aligned} P_j: \quad \boldsymbol{b}_1 x_1 + \boldsymbol{b}_2 x_2 + \cdots + \boldsymbol{b}_{16} x_{16} = \boldsymbol{b}_{17} \\ x_j \in \mathbb{Z}_-,\ x_i \in \mathbb{Z}_+,\ \text{ for } i \neq j, \end{aligned}$$

for $j = 1, 2, \cdots, 16$. We solved these systems via `lrs` and `LattE` [DeLoera et al. (2003)]. Then we have:

$$\begin{aligned} \boldsymbol{b}_{17} &= -\boldsymbol{b}_1 + \boldsymbol{b}_2 + \boldsymbol{b}_3 + \boldsymbol{b}_5 + \boldsymbol{b}_9 + \boldsymbol{b}_{16}, \\ \boldsymbol{b}_{17} &= \boldsymbol{b}_1 - \boldsymbol{b}_2 + \boldsymbol{b}_4 + \boldsymbol{b}_6 + \boldsymbol{b}_{10} + \boldsymbol{b}_{15}, \\ \boldsymbol{b}_{17} &= \boldsymbol{b}_1 - \boldsymbol{b}_3 + \boldsymbol{b}_4 + \boldsymbol{b}_7 + \boldsymbol{b}_{11} + \boldsymbol{b}_{14}, \end{aligned}$$

$$
\begin{aligned}
\boldsymbol{b}_{17} &= \boldsymbol{b}_2 + \boldsymbol{b}_3 - \boldsymbol{b}_4 + \boldsymbol{b}_8 + \boldsymbol{b}_{12} + \boldsymbol{b}_{13},\\
\boldsymbol{b}_{17} &= \boldsymbol{b}_1 - \boldsymbol{b}_5 + \boldsymbol{b}_6 + \boldsymbol{b}_7 + \boldsymbol{b}_{12} + \boldsymbol{b}_{13},\\
\boldsymbol{b}_{17} &= \boldsymbol{b}_2 + \boldsymbol{b}_5 - \boldsymbol{b}_6 + \boldsymbol{b}_8 + \boldsymbol{b}_{11} + \boldsymbol{b}_{14},\\
\boldsymbol{b}_{17} &= \boldsymbol{b}_3 + \boldsymbol{b}_5 - \boldsymbol{b}_7 + \boldsymbol{b}_8 + \boldsymbol{b}_{10} + \boldsymbol{b}_{15},\\
\boldsymbol{b}_{17} &= \boldsymbol{b}_4 + \boldsymbol{b}_6 + \boldsymbol{b}_7 - \boldsymbol{b}_8 + \boldsymbol{b}_9 + \boldsymbol{b}_{16},\\
\boldsymbol{b}_{17} &= \boldsymbol{b}_1 + \boldsymbol{b}_8 - \boldsymbol{b}_9 + \boldsymbol{b}_{10} + \boldsymbol{b}_{11} + \boldsymbol{b}_{13},\\
\boldsymbol{b}_{17} &= \boldsymbol{b}_2 + \boldsymbol{b}_7 + \boldsymbol{b}_9 - \boldsymbol{b}_{10} + \boldsymbol{b}_{12} + \boldsymbol{b}_{14},\\
\boldsymbol{b}_{17} &= \boldsymbol{b}_3 + \boldsymbol{b}_6 + \boldsymbol{b}_9 - \boldsymbol{b}_{11} + \boldsymbol{b}_{12} + \boldsymbol{b}_{15},\\
\boldsymbol{b}_{17} &= \boldsymbol{b}_4 + \boldsymbol{b}_5 + \boldsymbol{b}_{10} + \boldsymbol{b}_{11} - \boldsymbol{b}_{12} + \boldsymbol{b}_{16},\\
\boldsymbol{b}_{17} &= \boldsymbol{b}_4 + \boldsymbol{b}_5 + \boldsymbol{b}_9 - \boldsymbol{b}_{13} + \boldsymbol{b}_{14} + \boldsymbol{b}_{15},\\
\boldsymbol{b}_{17} &= \boldsymbol{b}_3 + \boldsymbol{b}_6 + \boldsymbol{b}_{10} + \boldsymbol{b}_{13} - \boldsymbol{b}_{14} + \boldsymbol{b}_{16},\\
\boldsymbol{b}_{17} &= \boldsymbol{b}_2 + \boldsymbol{b}_7 + \boldsymbol{b}_{11} + \boldsymbol{b}_{13} - \boldsymbol{b}_{15} + \boldsymbol{b}_{16},\\
\boldsymbol{b}_{17} &= \boldsymbol{b}_1 + \boldsymbol{b}_8 + \boldsymbol{b}_{12} + \boldsymbol{b}_{14} + \boldsymbol{b}_{15} - \boldsymbol{b}_{16}.
\end{aligned}
$$

Thus by Theorem 3.7, the number of elements in H is finite.

$2 \times 2 \times 2 \times 2$ tables with 2-marginals and a 3-marginal

Now we consider $2 \times 2 \times 2 \times 2$ tables with three 2-marginals and a 3-marginal as the simplicial complex on 4 nodes [12][13][14][234] and with 2 states on each node.

After removing redundant rows (using `cddlib`), $2 \times 2 \times 2 \times 2$ tables with 2-marginals and a 3-marginal has the 12×16 matrix A. Thus the semigroup is generated by 16 vectors in $\mathbb{Z}^{12}$ such that:

```
1 0 0 0 1 0 0 0 1 0 0 0 1 0 0 0
0 1 0 0 0 1 0 0 0 1 0 0 0 1 0 0
0 0 1 0 0 0 1 0 0 0 1 0 0 0 1 0
0 0 0 1 0 0 0 1 0 0 0 1 0 0 0 1
1 0 1 0 0 0 0 0 1 0 1 0 0 0 0 0
0 1 0 1 0 0 0 0 0 1 0 1 0 0 0 0
1 0 1 0 1 0 1 0 0 0 0 0 0 0 0 0
0 1 0 1 0 1 0 1 0 0 0 0 0 0 0 0
1 1 0 0 0 0 0 0 0 0 0 0 0 0 0 0
0 0 1 1 0 0 0 0 0 0 0 0 0 0 0 0
0 0 0 0 1 1 0 0 0 0 0 0 0 0 0 0
0 0 0 0 0 0 0 0 1 1 0 0 0 0 0 0
```

All of these vectors are extreme rays of the cone (verified via `cddlib`). The Hilbert basis of

the cone generated by these 16 vectors consists of these 16 vectors and two additional vectors

$$\boldsymbol{b}_{17} = (1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 0\ 0\ 0)^t, \qquad \boldsymbol{b}_{18} = (1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 0\ 1\ 1\ 1)^t.$$

Thus, $\boldsymbol{b}_{17}$, $\boldsymbol{b}_{18} \notin Q$. Then we set the system of linear equations such that:

$$\begin{gathered} \boldsymbol{b}_1 x_1 + \boldsymbol{b}_2 x_2 + \cdots + \boldsymbol{b}_{16} x_{16} = \boldsymbol{b}_{17} \\ x_1 \in \mathbb{Z}_-,\ x_i \in \mathbb{Z}_+,\ \text{for } i = 2, \cdots, 16. \end{gathered}$$

We solved the system via `lrs` and `CDD`. We noticed that this system has no real solution (infeasible). This means that

$$\boldsymbol{b}_{17} \notin (-\mathbb{N}\boldsymbol{a}_1) + \tilde{Q}_{(1)}.$$

Thus by Theorem 3.7, the number of elements in H is infinite.

Results on three-way tables

Results on the saturation of 3-DIPTP are summarized in Theorem 6.4 of [Ohsugi and Hibi (2006)]. They show that a normality (i.e., Q is saturated) or non-normality (i.e., Q is not saturated) of Q is not known only for the following three cases:

$$5 \times 5 \times 3, \quad 5 \times 4 \times 3, \quad 4 \times 4 \times 3.$$

All $2 \times J \times K$ tables are unimodular and hence saturated. This means that there is no hole in Q, and thus a $2 \times 2 \times 2$ example in [Irving and Jerrum (1994)] is not a hole. All $3 \times 3 \times J$ tables are saturated by the result of Sullivant (2004).

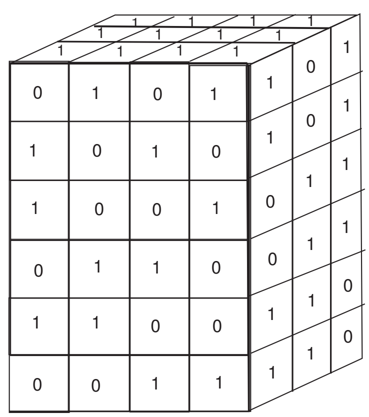

Figure 3: An example of $3 \times 4 \times 6$ table such that the given marginal condition is a hole of the semigroup.

For $3 \times 4 \times 6$ tables with 2-marginals, Vlach (1986) showed an example which has a table with nonnegative real entries, but does not have a table with nonnegative integer entries. This example can be found in Figure 3. Actually it is a particular example of Lemma 6.1 of

[Ohsugi and Hibi (2006)]. Aoki and Takemura (2003) presents a non-squarefree indispensable move $\boldsymbol{z} = \boldsymbol{z}^+ - \boldsymbol{z}^-$ of size $3 \times 4 \times 6$, where 2 appears both in the positive part $\boldsymbol{z}^+$ and the negative part $\boldsymbol{z}^-$. For this $\boldsymbol{z}$ there exist two standard coordinate vectors $\boldsymbol{e}_1, \boldsymbol{e}_2$ such that

$$\boldsymbol{u} = \boldsymbol{z}^+ - 2\boldsymbol{e}_1 \geq 0, \quad \boldsymbol{v} = \boldsymbol{z}^- - 2\boldsymbol{e}_2 \geq 0.$$

In this case Lemma 6.1 of [Ohsugi and Hibi (2006)] proves that $\boldsymbol{b} = A(\boldsymbol{u}+\boldsymbol{v})/2 \in Q_{\text{sat}}$ is a hole and this corresponds to Vlach's example.

Using Vlach's example, one can also show that $3\times4\times7$ tables and bigger tables have infinitely many holes. We take the example in Figure 3. Then we embed the table in a $3 \times 4 \times 7$ table. Then we put a single arbitrary positive integer c at just one place of the seventh 3×4 slice. This positive integer is uniquely determined by 2-marginals of the seventh slice alone (Table 1). Thus for each choice of c the beginning $3 \times 4 \times 6$ part remains to be a hole. Since c is arbitrary, $3 \times 4 \times 7$ table has infinite number of holes.

					sum
	c	0	0	0	c
	0	0	0	0	0
	0	0	0	0	0
sum	c	0	0	0	c

Table 1: the 7-th 3×4 slice is uniquely determined by its row and its column sums. c is an arbitrary positive integer.

We can generalize this idea as follows. Let A_1 denote the integer matrix corresponding to problem of a smaller size. Suppose that A for a larger problem can be written as a partitioned matrix

$$\begin{pmatrix} A_1 & 0 \\ 0 & A_2 \\ A_3 & A_4 \end{pmatrix},$$

where A_3 and A_4 are arbitrary. We consider the case that for A_1 there exists a hole. Now consider the semigroup associated with A_2. We assume that there exists infinite number of one-element fibers for the semigroup associated with A_2. This is usually the case, because the fibers on the extreme ray for A_2 is all one-element fibers, under the condition that A_2 does not contain more than one extreme rays in the same direction.

Under these assumptions consider the equation

$$\begin{pmatrix} t_1 \\ t_2 \\ t_3 \end{pmatrix} = \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \\ A_3 & A_4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix},$$

where t_1 is a hole for A_1, t_2 is any of the one-element fibers for A_2 and t_3 is chosen to satisfy the equation. Then $(t_1, t_2, t_3)^t$ is a hole for each t_2. Therefore there exist infinite number of holes for the larger problem.

Example 5.1. *Let A_1 be an integral matrix such that*

$$A_1 = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 3 & 4 \end{pmatrix}.$$

and let $A_2 = (1)$. From Example 2.6, H consists of only one element $\{t_1 = (1,2)^t\}$ and with A_2 we can find a family of infinite number of one-element fibers, namely $F_c := \{c\}$, where c is an arbitrary positive integer. Let $t_2 = c$. Then we have a matrix A such that:

$$A = \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 3 & 4 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

Note that $(t_1, t_2)^t = (1, 2, c)^t$ is a hole for each $t_2 = c$. Thus, since c is an arbitrary positive integer, there exist infinitely many holes for the semigroup generated by the columns of the matrix A.

6 Time complexity

In 2002, Barvinok and Woods (2003) introduced an algorithm to encode all integral vectors $\boldsymbol{b} \in \mathbb{Z}^d$ in Problem 1.1 as a *short rational generating function* in polynomial time when d and n are fixed (Lemma 6.3 stated below). However, in a sub-step of the algorithm they use the *Projection Theorem* which is not implementable at present. Thus, we do not know whether it is practical or not. From Lemma 6.3, we can show that the time complexity of computing H is polynomial time if we fix d and n (Corollary 6.5).

One might ask the time complexity of Problem 1.2. Using the results from [Barvinok (1994); Barvinok and Pommersheim (1999); Barvinok and Woods (2003)], we can prove that Problem 1.2 can be solved in polynomial time in fixed d and n (Theorem 6.1). In order to prove the theorem, we will use the *multivariate generating function* of a set $X \subset \mathbb{Z}^d$, $f(X; x)$. Namely, if $X \subset \mathbb{Z}^d$, define the generating function

$$f(X; x) = \sum_{s \in X} x^s,$$

where x^s denotes $x_1^{s_1} \cdots x_d^{s_d}$ with $s = (s_1, \ldots, s_d)$. If $X = P \cap \mathbb{Z}^d$ with fixed d, where P is a rational convex polyhedron, or if $X = Q$ with fixed d and n, then Barvinok (1994) and Barvinok and Woods (2003), respectively, showed that $f(X; x)$ can be written in the form of a polynomial-size sum of rational function of the form:

$$f(X; x) = \sum_{i \in I} \gamma_i \frac{x^{\alpha_i}}{\prod_{j=1}^{d} (1 - x^{\beta_{ij}})}. \quad (10)$$

Herein, I is a finite (polynomial size) index set and all the appearing data $\gamma_i \in \mathbb{Q}$ and $\alpha_i, \beta_{ij} \in \mathbb{Z}^d$ is of size polynomial. If a rational generating function $f(X; x)$ is polynomial size in the total bit size of inputs, then $f(X; x)$ is called a *short rational generating function*. As an example, if P is the one-dimensional polytope $[0, N]$, $N \in \mathbb{Z}_+$, then $f(P \cap \mathbb{Z}; x) = 1 + x + x^2 + \cdots + x^N$, $f(P \cap \mathbb{Z}; x)$ can be represented by a short rational generating function $(1 - x^{N+1})/(1 - x)$.

Theorem 6.1. *Suppose we fix d and n. There is a polynomial time algorithm in terms of the input size to decide whether the set of holes, H, for the semigroup, Q, generated by the columns of A is finite or not.*

Using the generating functions, we can show that the computation of fundamental holes for Q can be solved polynomial time if we fix d and n.

Theorem 6.2. *Suppose we fix d and n. Suppose Q is not saturated. The set of fundamental holes, H_0, can be encoded in a short rational generating function in time polynomial in terms of the input size.*

One notes that this algorithm outputs a generating function in the form of a short rational generating function. Therefore this does not return an explicit representation of H_0. However, if one wants to enumerate all elements in H_0, one can do the following: from the proof of Proposition 3.1, we have $H_0 \subset P \cap \mathbb{Z}^d$, where

$$P := \{x \in \mathbb{R}^d : x = \sum_{i=1}^{n} \delta_i \boldsymbol{a}_i,\ 0 \le \delta_i \le 1\}. \tag{11}$$

This shows that H_0 is finite and also gives a finite procedure to enumerate H_0:

- Compute the Hilbert basis B of $\text{cone}(\boldsymbol{a}_1, \ldots, \boldsymbol{a}_n) \cap L$.
- Check each $z \in B$ whether it is a fundamental hole or not, that is, compute $B \cap H_0$.
- Generate all nonnegative integer combinations of elements in $B \cap H_0$ that lie in $P \cap \mathbb{Z}^d$ and check for each such z whether it is a fundamental hole or not.

For more details, see [Hemmecke et al. (2007)].

Before proofs of Theorem 6.1 and Theorem 6.2, we would like to state lemmas from [Barvinok and Woods (2003)] and [Barvinok and Pommersheim (1999)].

Lemma 6.3 ((7.3) in [Barvinok and Woods (2003)])**.** *Suppose we fix d and n. Let $Q = Q(A)$. Then the generating function $f(Q; x)$ for the semigroup Q can be computed in polynomial time in terms of the input size as a short rational generating function in the form of* (10)*.*

Lemma 6.4 (Theorem 4.4 in [Barvinok and Pommersheim (1999)])**.** *Suppose we fix d and suppose $P \subset \mathbb{R}^d$ is a rational convex polyhedron. Then the generating function $f(P \cap \mathbb{Z}^d; x)$ can be computed in polynomial time in terms of the input size as a short rational generating function in the form of* (10)*.*

By Lemma 6.3 and Lemma 6.4, immediately, we have the following result.

Corollary 6.5. *Suppose we fix d and n. Let $Q = Q(A)$. Then the generating function $f(H; x)$ for the set of holes, $H := Q_{\text{sat}} \backslash Q$, can be computed in polynomial time in terms of the input size as a short rational generating function in the form of* (10).

Proof. Suppose we fix d and n. By Lemma 6.3, we can compute the generating function $f(Q; x)$ for the semigroup Q in polynomial time and by Lemma 6.4 we can compute the generating function $f(Q_{\text{sat}}; x)$ for the semigroup Q_{sat} in polynomial time. The generating function $f(H; x)$ for H is $f(Q_{\text{sat}}; x) - f(Q; x)$. $\square$

Using Corollary 6.5, we can prove Theorem 6.1.

Proof of Theorem 6.1. Suppose we fix d and n. First, we use Corollary 6.5 to compute the generating function, $f(H; x)$, for H in polynomial time in the form of (10). Let

$$f(H; x) = \sum_{i \in I} \gamma_i \frac{x^{\alpha_i}}{\prod_{j=1}^{d}(1 - x^{\beta_{ij}})}.$$

Then, we will do the following: First we choose $l \in \mathbb{Z}^d$ so that $\langle l, \beta_{ij} \rangle \neq 0$. We find such l in polynomial time by Lemma 2.5 in [Barvinok and Woods (2003)]. Let $l = (\lambda_1, \ldots, \lambda_d) \in \mathbb{Z}^d$. For $\tau > 0$, let $x_\tau = (\exp(\tau\lambda_1), \ldots, \exp(\tau\lambda_d))$ and let $\xi_{ij} = \langle l, \beta_{ij} \rangle$ and $\nu_i = \langle l, \alpha_i \rangle$. Then we apply the monomial substitution $x_i \to \exp(\tau\lambda_i)$. We can do this monomial substitution in polynomial time by Lemma 2.5 and Theorem 2.6 in [Barvinok and Woods (2003)]. Then

$$f(H; x_\tau) = \frac{1}{\tau^d} \left(\sum_{i \in I} \gamma_i \frac{\tau^d \exp(\tau\nu_i)}{\prod_{j=1}^{d}(1 - \exp(\tau\xi_{ij}))} \right).$$

Let

$$h_i(\tau) = \frac{\tau^d \exp(\tau\nu_i)}{\prod_{j=1}^{d}(1 - \exp(\tau\xi_{ij}))}$$

is a holomorphic function in a neighborhood of $\tau = 0$ and we take the Taylor expansion around $\tau = 0$ (i.e., we take the Laurent expansion around $\tau = 0$ for $h_i(\tau)/\tau^d$). The coefficients of the kth powers, where $0 \leq k \leq d - 1$, of the Taylor expansion of h_i are:

$$\frac{1}{\xi_{i1} \cdots \xi_{id}} \left(\sum_{l=0}^{k} \frac{\nu_i^l}{l!} \mathrm{td}_{k-l}(\xi_{i1}, \ldots, \xi_{id}) \right),$$

where $\mathrm{td}_l(\xi_{i1}, \ldots, \xi_{id})$ is a homogeneous polynomial of degree l and which is called the lth *Todd polynomial* in $\xi_{i1}, \ldots, \xi_{id}$ (see more details in Definition 5.1 in [Barvinok and Pommersheim (1999)]).

Now we claim that if the coefficients of negative powers of the Laurent expansion of $(\sum_{i \in I} h_i(\tau))/\tau^d$ are all canceled, then H has to be finite. We prove this by contradiction. Suppose H is infinite.

Then, since all coefficients of negative powers in the Laurent expansion are canceled, the sum of the coefficients of the constant terms:

$$\sum_{i \in I} \frac{\gamma_i}{\xi_{i1} \cdots \xi_{id}} \left(\sum_{l=0}^{d} \frac{\nu_i^l}{l!} \mathrm{td}_{d-l}(\xi_{i1}, \ldots, \xi_{id}) \right) \tag{12}$$

must be equal to the number of elements in H when we send $\tau \to 0$ ((5.2) [Barvinok and Pommersheim (1999)]). Thus, the sum of the coefficients of the constant terms in (12) must be equal to infinity. Since I is a finite index set, a coefficient of the constant term in the Laurent expansion of some rational function must be infinite. However, the Todd polynomials are polynomials in $\mathbb{C}$ so it is impossible. Thus we reach a contradiction.

Conversely, it is obvious that if the coefficients of negative powers of the Laurent expansion of $(\sum_{i \in I} h_i(\tau))/\tau^d$ are not canceled, then H is infinite.

Therefore we will have to check all coefficients of the kth powers, where $0 \leq k \leq d-1$, of the Taylor expansion of each $h_i(\tau)$. Since we have the polynomial size index set I and we have to only check d coefficients for each rational function, this computation can be done in polynomial time. $\square$

Now we would like to discuss the *intersection algorithm*, which we need to encode H_0 in a short rational generating function in polynomial time in fixed d and n.

Lemma 6.6 (Theorem 3.6 in Barvinok and Woods (2003))**.** *Let S_1, S_2 be finite subsets of $\mathbb{Z}^d$, for fixed d. Let $f(S_1; x)$ and $f(S_2; x)$ be their generating functions, given as short rational generating functions with at most k binomials in each denominator. Then there exist a polynomial time algorithm, which, given $f(S_i; x)$, computes*

$$f(S_1 \cap S_2; x) \quad = \quad \sum_{i \in I} \gamma_i \cdot \frac{x^{u_i}}{(1 - x^{v_{i1}}) \cdots (1 - x^{v_{is}})}$$

with $s \leq 2k$, where the γ_i are rational numbers, u_i, v_{ij} nonzero integers, and I is a polynomial-size index set.

The essential step in the *intersection algorithm* is the *Hadamard product* [Definition 3.2 in Barvinok and Woods (2003)]. Using Lemma 6.6, we can compute the union of s sets in $\mathbb{Z}^d$ in polynomial time for fixed d and s.

We now give a proof of Theorem 6.2.

Proof of Theorem 6.2. Suppose Q is not saturated. Using Lemma 6.4, we compute the generating function $f(P \cap \mathbb{Z}^d; x)$ in polynomial time, where P is given in (11). Note that there are 2^n points in $P \cap Q$, namely $\{x \in Q : x = \sum_{i=1}^n \xi_i \boldsymbol{a}_i,\ \xi_i \in \{0, 1\}\}$. So we can enumerate all points in $P \cap Q$ in constant time. Let $\bar{H} = (P \cap \mathbb{Z}^d) \setminus (P \cap Q)$. Its generating function $f(\bar{H}; x)$ is $f(P \cap \mathbb{Z}^d; x) - f(P \cap Q; x)$ and it can be computed in polynomial time. Note that $H_0 = \bar{H} \setminus ((\bar{H} + (P \cap Q)) \cap \bar{H})$ from the definition of H_0 and $H_0 \subset (P \cap \mathbb{Z}^d) \setminus (P \cap Q)$.

We compute the generating function for $(\bar{H} + (P \cap Q))$ by the following: let $(P \cap Q) \setminus \{0\} = \{z_1, \cdots, z_{2^n-1}\}$. For each $i = 1, 2, \ldots, 2^n - 1$, let $g_i(x) := x^{z_i} \cdot f(\bar{H}; x)$ which is the generating function for the set $z_i + \bar{H}$. Since $2^n - 1$ is a constant (we are fixing n as a constant), applying Lemma 6.6 we can compute the generating function for the union of $z_i + \bar{H}$ in polynomial time. Since $\bar{H}$ and $(P \cap Q)$ are finite we are done. $\square$

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